Random symmetric matrices with a constraint: The spectral density of random impedance networks

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(Received 11 November 2002; published 10 April 2003)

We derive the mean eigenvalue density for symmetric Gaussian random $N\times N$ matrices in the limit of large N, with a constraint implying that the row sum of matrix elements should vanish. The result is shown to be equivalent to a result found recently for the average density of resonances in random impedance networks [Y.V. Fyodorov, J. Phys. A **32**, 7429 (1999)]. In the case of banded matrices, the analytical results are compared with those extracted from the numerical solution of Kirchhoff equations for quasi-one-dimensional random impedance networks.

DOI: 10.1103/PhysRevE.67.047101 PACS number(s): 02.50.−r

The study of random matrices [1] has provided insight into many physical problems, both in the quantum and in the classical domain. For example, random matrices have been very successfully used to model statistical properties of disordered conductors and of highly excited classically chaotic quantum systems [2]. In the classical domain random matrices arise, for instance, in the context of diffusion in random, directed environments (see, for example, Ref. [3], and references therein). In most of those applications the random-matrix elements obeyed symmetry requirements where appropriate, and were otherwise taken to be independently distributed random variables.

There are cases, however, where constraints on the matrix elements must be considered. Such constraints generate correlations between the latter; examples are electron hopping in amorphous semiconductors (see Ref. [4], and references cited therein), random impedance networks [5–7] (see Ref. [8] for review and Ref. [9] for applications), and random master equations [10]: in these cases, the random matrices obey the constraints that the row sums of matrix elements should be zero. This condition implies correlations between diagonal and off-diagonal matrix elements. In Refs. [4,6,10], the average spectral density and spectral fluctuations of three different random-matrix ensembles of this type were calculated, using the method of replicas and the supersymmetry approach.

In this paper, we derive the average density of eigenvalues of a suitably modified ensemble of symmetric Gaussian random matrices. The aim is twofold. First, we wish to show that for full matrices the density obtained is equivalent to that found in Refs. [6,10]. Our second aim is to derive the corresponding result for banded matrices, and to compare it with the results of the numerical solution of the Kirchhoff equations on random impedance networks with quasi-one-dimensional topology.

I. FORMULATION OF THE PROBLEM

We consider an ensemble of $N\times N$ random matrices $\mathbf{M}$ with matrix elements

$$M_{mn}=J_{mn}-\delta_{mn}\sum_{l=1}^{N}J_{ml}, \tag{1}$$

where $\mathbf{J}$ is a real symmetric $N\times N$ matrix with random entries. Such ensembles have been considered in Refs. [6,7,10,11]. In Ref. [10], $\mathbf{M}$ was used to model a random transition matrix for a model describing glassy relaxation, $\partial_t u(t)=-\mathbf{M}u(t)$, with matrix elements J_{mn} distributed independently (subject to the constraint $J_{mn}=J_{nm}$) according to

$$P(J_{mn})=(p/N)\,\delta(J_{mn}-1/p)+(1-p/N)\,\delta(J_{mn}). \tag{2}$$

The form of Eq. (1) yields $\Sigma_n M_{mn}=0$, implying probability conservation in the problem considered in Ref. [10]. The average density $d(\lambda)$ of eigenvalues λ,

$$d(\lambda)=N^{-1}\langle \mathrm{tr}\,\delta(\mathbf{M}-\lambda\mathbf{1})\rangle, \tag{3}$$

was calculated in the limit of large N and p, using the method of replicas. $\langle\cdots\rangle$ is an average over the ensemble defined by $P(J_{mn})$.

In Ref. [6], eigenvalues of matrices $\mathbf{M}$ with elements similar to Eq. (1) were shown to model resonance frequencies in random impedance networks [5], with $J_{mn}=J_{nm}$ and

$$P(J_{mn})=\tfrac{1}{2}\,\delta(J_{mn}-1)+\tfrac{1}{2}\,\delta(J_{mn}+1). \tag{4}$$

In Refs. [6,7] the average density of resonance frequencies was calculated, in the limit of large N, using a variant of the supersymmetry technique. It was found that the result agrees

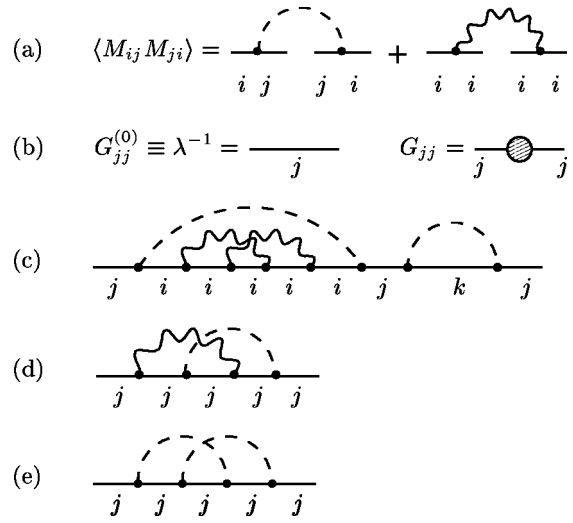

FIG. 1. (a) Contraction $\langle M_{ij}M_{ji}\rangle$: the dashed line carries a factor of $1/N$ and the wavy line carries a factor of unity. The second term contributes only when $i=j$. (b) Graphical representation of $G_{jj}^{(0)}\equiv E^{-1}$ and G_{jj}, the jth diagonal matrix element of the average resolvent **G**. (c) Example of a diagram contributing to G_{jj}. Internal indices are summed over (i and k). (d) and (e) Diagrams of higher order in N^{-1}.

[12] with that of Ref. [10], up to a scale factor (related to p) and a rigid shift in λ [related to the fact that $J_{mn}\geqslant 0$ in Eq. (2)].

In the following, we calculate the ensemble-averaged density of eigenvalues of **M**, treating both off-diagonal and diagonal entries of **J** as independent, identically distributed Gaussian real variables. The corresponding symmetric random matrix **J** belongs to the Gaussian orthogonal ensemble [1] with joint probability density

$$P(\mathbf{J})d\mathbf{J}\propto \exp-(1/4\sigma^2)\mathrm{tr}\,\mathbf{J}^2 d\mathbf{J}. \tag{5}$$

The average density of eigenvalues E of such matrices in the limit $N\gg 1$ is given by the semicircular law $d(E)$ $=(2\pi\sigma^2 N)^{-1}\sqrt{4\sigma^2 N-E^2}$ for $|E|\leqslant 2(\sigma^2 N)^{1/2}$ and zero otherwise [13]. In the following, we ask how imposing a ''constraint'' (that the row sum of matrix elements should be zero) modifies the mean eigenvalue density of **M** with respect to that of **J**. In the limit of large N, the problem may be solved using diagrammatic perturbation theory (see, for instance Ref. [3], and references cited therein), as shown below.

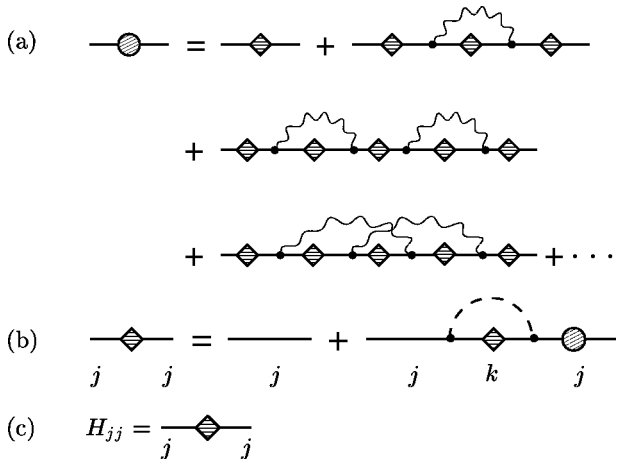

FIG. 2. (a) and (b) Self-consistent equations for the average of the resolvent. (c) Graphical representation of the auxiliary functions H_{jj}.

Our results may be summarized as follows. In the limit of large N, the averaged eigenvalue density for **M** coincides with the result derived in Refs. [6,7]. In this limit, correlations between diagonal and off-diagonal matrix elements are found to be irrelevant, and the result can be understood in terms of an averaged Pastur equation [14]. Furthermore, we have also considered the case of banded symmetric random matrices. This case is of interest for random impedance networks with quasi-one-dimensional topology [5,11]. Our results are in good agreement with those of exact numerical solutions of the Kirchhoff equations for such networks.

II. METHOD

The average eigenvalue density may be obtained from the trace of the averaged resolvent $\mathbf{G}=\langle(E\mathbf{1}-\mathbf{M})^{-1}\rangle$,

$$d(E)=-(N\pi)^{-1}\mathrm{Im}\,\mathrm{tr}\,\mathbf{G}. \tag{6}$$

Expanding the resolvent, Wick's theorem may be employed for performing the average, using

$$\begin{aligned}\langle M_{ij}M_{mn}\rangle=\sigma^2[&(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})-\delta_{mn}(\delta_{im}\delta_{jl}+\delta_{il}\delta_{jm})\\&-\delta_{ij}(\delta_{im}\delta_{kn}+\delta_{in}\delta_{km})+\delta_{ij}\delta_{mn}(N\delta_{im}+1)].\end{aligned} \tag{7}$$

In the limit of large N, adopting the scaling $\sigma^2=N^{-1}$, Eq. (7) can be simplified to

$$\langle M_{ij}M_{mn}\rangle\simeq N^{-1}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})+\delta_{ij}\delta_{mn}\delta_{im}. \tag{8}$$

It is convenient to keep track of the contributions with the help of diagrams. To this end, a graphical representation of the contraction (8) is introduced in Fig. 1(a). The averaged resolvent **G** turns out to be a diagonal matrix. Figure 1(b) defines a graphical representation for the diagonal elements G_{jj} of **G**. Terms contributing to G_{jj} are shown in Fig. 1(c–e). One observes that, in the limit of large N, only diagrams with no intersections between dashed or between dashed and wavy lines contribute. Thus, to leading order in N, the contribution (c) in Fig. 1 must be considered, but not (d) or (e), for example.

All diagrams contributing to the resolvent to leading order in N^{-1} may be summed, resulting in a set of two coupled equations shown in Figs. 2(a) and 2(b). The diagonal elements G_{jj} and H_{jj} are independent of j and denoted by G and H in the following. The first equation [Fig. 2(a)] contains an infinite sum of diagrams. This sum may be performed exactly. It is denoted by $g(H^{-1})$, where

$$g(z)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}dJ\,\frac{\exp(-J^2/2)}{z-J}. \tag{9}$$

The result for the resolvent scales with σ^2 and N as $(\sigma^2 N)^{-1/2} g[H^{-1}(\sigma^2 N)^{1/2}]$. Consequently, the equations shown in Fig. 2(a,b) imply the following self-consistent equation:

$$G=(\sigma^2 N)^{-1/2} g[(E-G)(\sigma^2 N)^{1/2}], \tag{10}$$

equivalent to Eq. (51) in Ref. [6] for the average density of resonances in an ensemble of highly connected random impedance networks [15].

The form of the simplified contraction (8) implies an interpretation of the result (10) in terms of an averaged Pastur equation: consider a random matrix $\mathbf{M}=\mathbf{J}+\mathbf{V}$, where $\mathbf{J}$ is distributed according to Eq. (5), and $\mathbf{V}$ is a diagonal matrix with Gaussian random entries v_k with zero mean and unit variance, independent of J_{mn}. For a given realization of $\mathbf{V}$, Pastur's equation [14] is $\mathbf{G}=(E\mathbf{1}-\mathbf{V}-\mathbf{G})^{-1}$ (see also Ref. [16], and references therein). In this equation $\mathbf{G}$ is the $\mathbf{J}$-averaged resolvent, keeping $\mathbf{V}$ fixed. One obtains Eq. (10) after observing that $\mathbf{G}$ is diagonal, by averaging over the matrix elements of $\mathbf{V}$. This interpretation implies that, in the limit of large N, the correlations between diagonal and off-diagonal matrix elements of $\mathbf{M}$ [as seen in Eqs. (1) and (7)] are irrelevant.

The above procedure is easily extended to the case of banded matrices, also of interest in random impedance networks [5,11]. In the banded case, $\langle J_{mn}J_{kl}\rangle=\sigma^2(\delta_{mk}\delta_{nl}+\delta_{ml}\delta_{nk})$ [which follows from Eq. (5)] is replaced by

$$\langle J_{mn}J_{kl}\rangle=\sigma^2(|m-n|)(\delta_{mk}\delta_{nl}+\delta_{ml}\delta_{nk}). \tag{11}$$

The function $\sigma^2(x)$ is given by

$$\sigma^2(x)=\begin{cases}\sigma^2 & \text{for } 0\leqslant x\leqslant b/2\\ 0 & \text{otherwise.}\end{cases} \tag{12}$$

The bandwidth of $\mathbf{J}$ is thus b. In the limit of large N and large b, the spectral density is given by a slight modification of Eq. (10),

$$G=(\sigma^2 b)^{-1/2} g[(E-G)(\sigma^2 b)^{1/2}]. \tag{13}$$

Diagrammatically, the necessary changes are most easily derived by letting $\sigma^2=N^{-1}$ and assuming that $b=BN$ (with fixed B). Then the wavy line in Fig. 1(a) acquires a factor of B. Furthermore, the dashed line in Eq. (8) also acquires a factor of B. Hence, the self-consistency equation in the banded case becomes (13). Let us also note that the same formula is actually valid not only for $b\sim N$, but more generally for $1\ll b\ll N$.

This result implies that densities for different values of b can be scaled on one single curve by plotting $\sqrt{b}d(E/\sqrt{b})$. In Fig. 3, solutions of Eq. (13) for $\sigma=1$ are compared with results of exact diagonalizations of random matrices with $N=500,1000$, and $b=50,100$. We observe a very good agreement. The results confirm that, for large N and b, the average density of eigenvalues is independent of N, and that it scales with b as expected.

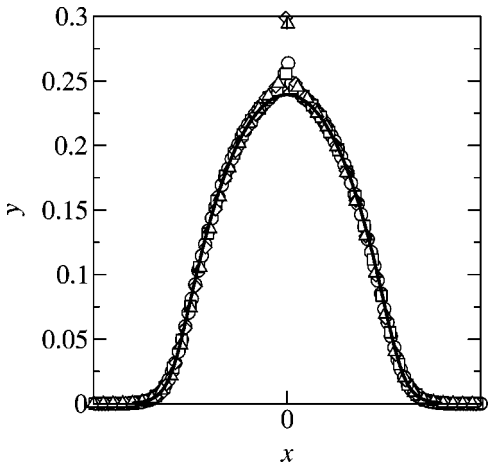

FIG. 3. Density of eigenvalues of exact diagonalizations of random matrices of the form (1), (5) for $\sigma=1$, $N=500,1000$, and $b=50,100$ (symbols), together with the prediction (13) (solid line).

In the following, we show that Eq. (13) also describes the density of resonances for certain random impedance networks with a large, but finite connectivity.

III. RANDOM IMPEDANCE NETWORKS

Random networks of complex impedances are currently used to model electrical and optical properties of disordered inhomogeneous media [8]. The most common situation is that of a binary composite medium, modeled by attributing a random conductance to each bond $(\mathbf{x},\mathbf{y})$ of a lattice, according to the binary law:

$$\sigma_{\mathbf{x},\mathbf{y}}=\begin{cases}\sigma_0 & \text{with probability } p\\ \sigma_1 & \text{with probability } q=1-p.\end{cases} \tag{14}$$

The homogeneous Kirchhoff equations for the electric potentials,

$$\sum_{\mathbf{y}} \sigma_{\mathbf{x},\mathbf{y}}(V_{\mathbf{y}}-V_{\mathbf{x}})=0, \tag{15}$$

can be recast as

$$(\Delta_Q-\lambda\Delta)V=0, \tag{16}$$

with $\lambda=\sigma_0/(\sigma_0-\sigma_1)$, and

$$(\Delta V)_{\mathbf{x}}=\sum_{\mathbf{y}(\mathbf{x})}(V_{\mathbf{y}}-V_{\mathbf{x}}),\quad (\Delta_P V)_{\mathbf{x}}=\sum_{\mathbf{y}\in P(\mathbf{x})}(V_{\mathbf{y}}-V_{\mathbf{x}}),$$
$$(\Delta_Q V)_{\mathbf{x}}=\sum_{\mathbf{y}\in Q(\mathbf{x})}(V_{\mathbf{y}}-V_{\mathbf{x}}), \tag{17}$$

where $\mathbf{y}(\mathbf{x})$ are all the sites connected to site $\mathbf{x}$, whereas $\mathbf{y}\in P(\mathbf{x})$ $(\mathbf{y}\in Q(\mathbf{x}))$ are those connected by a conductance σ_0 (σ_1), so that $\Delta=\Delta_P+\Delta_Q$. Resonances appear as nontrivial solutions to Eq. (16), for $0<\lambda<1$.

An efficient algorithm allowing for an exact determination of all the resonances of a finite sample has been devel-

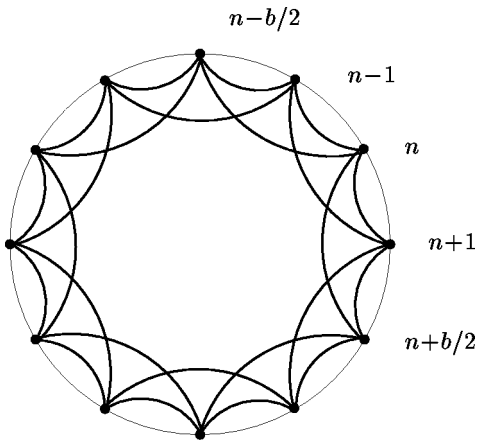

FIG. 4. Topology of a (quasi) one-dimensional lattice with periodic boundary conditions. Each site n is connected to its $b=4$ neighbors.

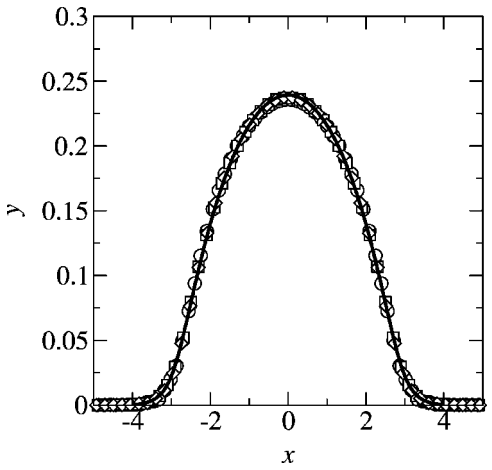

FIG. 5. Plot of the density of resonances for a quasi-one-dimensional random impedance network (see text). Data for three different ranges ($b=60$, 100, and 120) (symbols) are compared with the theoretical prediction (13).

oped in Ref. [5]. We have adapted this algorithm to the simplest geometry allowing for long-ranged bonds. Each site n of a very long chain is connected to its b neighbors ($n-b/2,\ldots,n-1,n+1,\ldots,n+b/2$), as shown in Fig. 4. Periodic boundary conditions are assumed. For definiteness, we choose $p=q=1/2$, so that all the resonances are expected to be located at $\lambda=1/2$ in the $b\to\infty$ limit.

Our numerical results are shown in Fig. 5, for very long periodic chains with ranges $b=60$, 100, and 120. For each value of b, we have accumulated a number of resonances of order 10^7. After rescaling the resonances according to $\lambda=(1+E)/2$ the density is given by Eq. (13) with $\sigma=1$. A very satisfactory quantitative agreement with the theoretical prediction is observed.

ACKNOWLEDGMENTS

Discussions with J. T. Chalker are gratefully acknowledged. Further, support from Vetenskapsrådet, EU network QTRANS (J.S. and B.M.), and EPSRC Research Grant No. GR/13838/01 (Y.V.F) is acknowledged with thanks.

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